Calculus Questions

3.73 Find the derivative & simplify

(a) $u(x) = \sin(\operatorname{arccot}(x) - \pi)$.

Solution:

$$u'(x) = \cos(\operatorname{arccot}(x) - \pi) \cdot \frac{-1}{1+x^2}$$

$$= \frac{+\cos(\operatorname{arccot}(x))}{1+x^2}$$

Side calculation: What is $\cos(\operatorname{arccot}(x))$?

Let $\theta = \operatorname{arccot}(x) \Rightarrow \cot\theta = x = \frac{x}{1}$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{x}{1}$$

$$\Rightarrow \cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{x}{\sqrt{1+x^2}}$$

$$\cos(\operatorname{arccot}(x)) = \frac{x}{\sqrt{1+x^2}}.$$

$$\Rightarrow u'(x) = \frac{+\cos(\operatorname{arccot}(x))}{1+x^2}$$

$$= \frac{\left(\frac{x}{\sqrt{1+x^2}}\right)}{1+x^2} = \boxed{\frac{x}{(1+x^2)^{3/2}}}.$$

(b) $F(x) = \tan(x)\sec(4^{3x})$

$$\Rightarrow F'(x) = \sec^2(x)\cdot\sec(4^{3x}) + \tan(x)\cdot\sec(4^{3x})\tan(4^{3x})\cdot(4^{3x})'$$

$$= \sec^2(x)\sec(4^{3x}) + \tan(x)\sec(4^{3x})\tan(4^{3x})\cdot(\ln 4)4^{3x}\cdot 3$$

$$\Rightarrow \boxed{F'(x) = \sec^2(x)\sec(4^{3x}) + 3(\ln 4)\tan(x)\sec(4^{3x})\tan(4^{3x})4^{3x}}$$

(c) $$h(\beta) = \frac{\beta\sqrt{\beta}\sqrt{4\beta^3} + 1}{\beta^3 - \sqrt{\beta^3}}$$

$$h(\beta) = \frac{\beta^1 \beta^{1/2}\, 2\beta^{3/2} + 1}{\beta^3 - \beta^{3/2}}$$

$$= \frac{2\beta^3 + 1}{\beta^3 - \beta^{3/2}}$$

$$h'(\beta) = \frac{Ho\ dHi - Hi\ dHo}{Ho\ Ho}$$

$$= \frac{(\beta^3 - \beta^{3/2})(6\beta^2) - (2\beta^3+1)(3\beta^2 - \frac{3}{2}\beta^{1/2})}{(\beta^3 - \beta^{3/2})^2}$$

$$= \frac{6\beta^5 - 6\beta^{7/2} - [6\beta^5 - 3\beta^{7/2} + 3\beta^2 - \frac{3}{2}\beta^{1/2}]}{[\beta^{3/2}(\beta^{3/2} - 1)]^2}$$

$$= \frac{\cancel{6\beta^5} - 6\beta^{7/2} - \cancel{6\beta^5} + 3\beta^{7/2} - 3\beta^2 + \frac{3}{2}\beta^{1/2}}{\beta^3(\beta^{3/2} - 1)^2}$$

$$= \frac{-3\beta^{7/2} - 3\beta^2 + \frac{3}{2}\beta^{1/2}}{\beta^3(\beta^{3/2} - 1)}$$

$$= \frac{3\beta^{1/2}\left(-\beta^{3} - \beta^{3/2} + \frac{1}{2}\right)}{\beta^{3}\left(\beta^{3/2} - 1\right)}$$

$$= \boxed{\frac{3\left(-\beta^{3} - \beta^{3/2} + \frac{1}{2}\right)}{\beta^{5/2}\left(\beta^{3/2} - 1\right)}}$$

(d) $L(t) = \ln(At)\arctan(Ae^{t^2})$, where A is a constant.

$$L'(t) = \left(\frac{1}{At} \cdot A\right)\arctan(Ae^{t^2}) + \ln(At) \cdot \frac{1}{1+(Ae^{t^2})^2} \cdot (Ae^{t^2})'$$

$$\Rightarrow L'(t) = \frac{1}{t}\arctan(Ae^{t^2}) + \frac{\ln(At)}{1+A^2e^{2t^2}} \cdot \left[A(2t)e^{t^2}\right]$$

$$\Rightarrow \boxed{L'(t) = \frac{\arctan(Ae^{t^2})}{t} + \frac{2At\ln(At)e^{t^2}}{1+A^2e^{2t^2}}}$$

Implicit Differentiation

Given a relation that is not a function, we can still use the derivative to find slopes & rates of change. How?: pretend y is a function of x, and just differentiate both sides of the equation.

Example (3.62 b) Find $\frac{dy}{dx}$.

$$y^3 + 3yx^2 + x^5 = xy^2.$$

Take $\frac{d}{dx}$ of both sides, thinking that y is a fcn of x:

$$3y^2\left(\frac{dy}{dx}\right) + 3\frac{dy}{dx}\cdot x^2 + 3y(2x) + 5x^4$$

$$= 1\cdot y^2 + x\left(2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left[3y^2+3x^2\right] + 6xy + 5x^4 = y^2 + 2xy\frac{dy}{dx}$$

$$-2xy\frac{dy}{dx} \qquad -2xy\frac{dy}{dx}$$

$-6xy \quad -5x^4 \qquad -6xy-5x^4$

$\Rightarrow$

$$\frac{dy}{dx}\left[3y^2+3x^2-2xy\right]=y^2-6xy-5x^4$$

$$\Rightarrow \quad \frac{dy}{dx} = \boxed{\frac{y^2-6xy-5x^4}{3y^2+3x^2-2xy}}.$$

3.62 c Find $\frac{dy}{dx}$:

$$xe^y = ye^x + 4y$$

Take $\frac{d}{dx}$ of both sides:

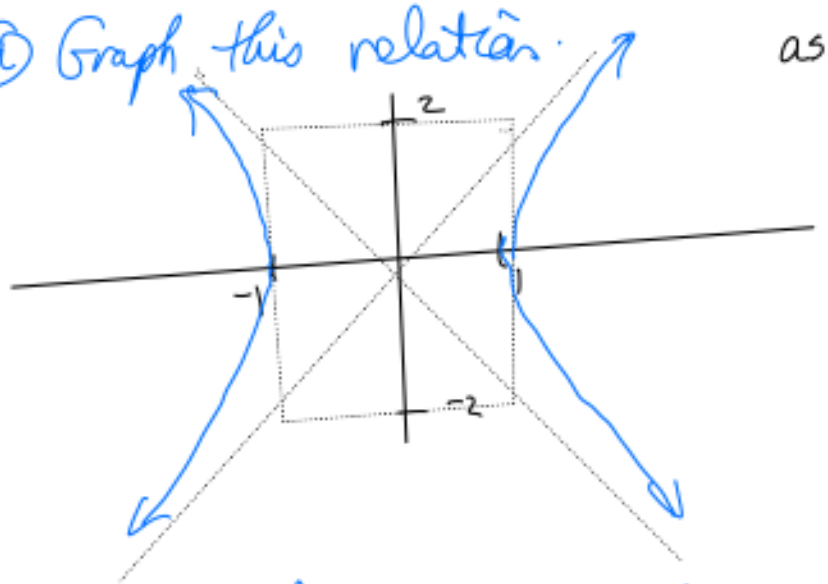
$$1\cdot e^y + x\cdot e^y\cdot\frac{dy}{dx} = \frac{dy}{dx}\cdot e^x + ye^x + 4\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[xe^y-e^x-4\right] = ye^x - e^y$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{ye^x-e^y}{xe^y-e^x-4}}.$$

3.61 Consider the hyperbola $x^2 - \frac{y^2}{4} = 1$.

(a) Graph this relation.



asymptotes

$$x^2 - \frac{y^2}{4} = 0$$

$$\left(x - \frac{y}{4}\right)\left(x + \frac{y}{4}\right) = 0$$

$$y = 4x, -4x$$

(b) Find $\frac{dy}{dx}$: $\quad x^2 - \frac{y^2}{4} = 1$

Take $\frac{d}{dx}$ of both sides:

$$2x - \frac{1}{4} 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2x = \frac{1}{2} y \frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{4x}{y}}$$

(c) Eqn of tangent line at $\left(\frac{5}{3}, \frac{-8}{3}\right)$.
$\qquad\qquad\qquad\qquad\qquad\qquad x \quad\; y$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4\left(\frac{5}{3}\right)}{-\frac{8}{3}} = -\frac{20}{8} = -\frac{5}{2}$$

$$\Rightarrow \quad y - y_0 = m(x - x_0)$$

$$y + \frac{8}{3} = \frac{-5}{2}\left(x - \frac{5}{3}\right)$$

$$= \frac{-5}{2}x + \frac{25}{6}$$

$$y = \frac{-5}{2}x + \underbrace{\frac{25}{6} - \frac{16}{6}}_{\frac{9}{6} = \frac{3}{2}}$$

$$\Rightarrow \boxed{y = \frac{-5}{2}x + \frac{3}{2}}$$